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New Trends in Quantum Field Theory

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Constrained KP Hierarchies: Darboux-Bäcklund Solutions and Additional Symmetries [1]

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Abstract

We illustrate the basic notions of *additional non-isospectral symmetries* and their interplay with the discrete *Darboux-Bäcklund transformations* of integrable systems at the instance of *constrained Kadomtsev-Petviashvili* (cKP) integrable hierarchies. As a main application we present the solution of discrete multi-matrix string models in terms of Wronskian τ-functions of graded $SL(m,1)$ cKP hierarchies.

1. Introduction. KP as an Arch-Type Integrable System

Integrable systems (for the basics, see refs. [1, 2, 3]) constitute an outstanding branch of theoretical physics since they describe a vast variety of fundamental non-perturbative phenomena ranging from $D = 2$ (space-time dimensional) nonlinear soliton physics and planar statistical mechanics to string and membrane theories in high-energy elementary particle physics. It turns out that, under plausible assumptions, a variety of physically interesting theories in higher space-time dimensions can be reformulated as lower-dimensional ($D = 2$) integrable models which in the same time possess *infinite-dimensional symmetries* and thus, as a rule, being integrable (see, especially, the recent developments [4] related with integrability of Seiberg-Witten effective low-energy theory of (extended) supersymmetric gauge theories).

Among the various infinite-dimensional symmetry groups and algebras playing rôle in integrable field theories, a particularly distinguished place belongs to the Lie algebra $W_{1+\infty}$ [5] (specific "large N limit" of Zamolodchikov's W_N conformal algebras [6], isomorphic to the Lie algebra of all purely differential operators on the circle). It contains (together with its supersymmetric extension) all previously known infinite-dimensional symmetry algebras – Virasoro and Kac-Moody. Also, it is precisely a Lie-algebraic deformation of the infinite-dimensional generalization of the Virasoro algebra – the algebra of area-preserving diffeomorphisms.

Recently $W_{1+\infty}$ symmetries attracted broad interest as they appeared naturally as inherent structures of models in different areas of theoretical physics : theory of black holes and space-time singularities, two-dimensional quantum gravity, nonlinear evolution equations in higher dimensions, self-dual gravity, $N = 2$ superstring theory (refs. (a)–(d) in [7]), quantum Hall effect [8]. All listed models possess, in one form or in another, *exactly soluble* features, which naturally suggest an intimate connection of *integrability* with $W_{1+\infty}$ algebra. This claim may, furthermore, be substantiated by the realization that $W_{1+\infty}$ algebra is a subalgebra of

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the algebra $\Psi\mathcal{DO}$ of arbitrary pseudo-differential operators on the circle. Already since the pioneering papers of Adler-Kostant-Symes and of the Faddeev's school [9] it was realized that $\Psi\mathcal{DO}$ forms the foundation of completely integrable systems. In fact, it turns out that most known integrable models (i.e. those admitting Lax or "zero curvature" representation) can be associated with specific coadjoint orbits of various subalgebras of $\Psi\mathcal{DO}$ or with different Hamiltonian reductions thereof [1].

The generic integrable system based on $\Psi\mathcal{DO}$ symmetry algebra is the Kadomtsev-Petviashvili (**KP**) integrable hierarchy [10, 11] of soliton nonlinear evolution equations. Its name derives from the fact that **KP** hierarchy contains the $D = 2+1$ dimensional nonlinear soliton **KP** equation which appeared originally in plasma physics. In the last few years the main interest towards **KP** hierarchy originates from its deep connection with the statistical-mechanical models of random matrices ((multi-)matrix models) providing non-perturbative discretized formulation of string theory [12].

The purpose of the present talk is to provide a brief discussion of the basic notions of *additional non-isospectral symmetries* [13, 11] and their interplay with the discrete *Darboux-Bäcklund transformations* [14] of integrable systems within the **KP** integrable hierarchy (being and arch-type integrable system, as pointed out above) and its various constrained versions (cKP hierarchies) relevant in discrete multi-matrix string models. Furthermore, we show how to obtain the solution of the latter in terms of Wronskian τ-functions of the cKP hierarchies.

2. KP Hierarchy: Pseudo-Differential Operator Formalism and Hamiltonian Structures (W-Algebras)

We describe the (generalized) **KP** integrable hierarchy in the language of pseudo-differential operators (for a background, see [11]). The main object is the pseudo-differential Lax operator L subject to an infinite set of evolution equations:

$$L = D^m + \sum_{j=0}^{m-2} v_j D^j + \sum_{i\geq 1} u_i D^{-i} \qquad , \qquad \frac{\partial}{\partial t_l} L = \left[L^{\frac{l}{m}}_{(+)} \, , \, L \right] \tag{1}$$

Here, the coefficients of L are (smooth) functions of $x \equiv t_1$ and the higher time-evolution parameters $t_2, t_3, \ldots$; $D \equiv \partial_x$, whereas the subscripts $(\pm)$ denote purely differential (purely pseudo-differential) part of the corresponding pseudo-differential operators. The flows $\frac{\partial}{\partial t_l}$ in (1) commute (as vector fields on the space of Lax operators (1)) among themselves which expresses the integrability of the **KP** system.

Within the Sato-Wilson dressing operator formalism, with the following dressing expression for the generalized **KP** Lax operator (1):

$$L = W D^m W^{-1} \qquad , \qquad W \equiv 1 + \sum_{i\geq 1} w_i D^{-i} \tag{2}$$

the evolution equations for L are equivalent to:

$$\frac{\partial}{\partial t_l} W = - \left(W D^l W^{-1} \right)_- W \tag{3}$$

In what follows we shall also need the important notions of *(adjoint) eigenfunctions* and *(adjoint) Baker-Akhiezer* functions. The function Φ (Ψ) is called (adjoint) eigenfunction for the Lax operator L satisfying Sato's flow equations (1) if its flows are given by expression:

$$\frac{\partial \Phi}{\partial t_l} = L^{\frac{l}{m}}_{(+)} \Phi \qquad ; \qquad \frac{\partial \Psi}{\partial t_l} = - \left(L^* \right)^{\frac{l}{m}}_{(+)} \Psi \tag{4}$$

for the infinite many times t_l [4]. If, in addition, an (adjoint) eigenfunction satisfies the spectral equation:

$$L\psi(\lambda) = \lambda\psi(\lambda) \;\; , \;\; \frac{\partial}{\partial t_l}\psi(\lambda) = L^{\frac{l}{m}}_{(+)}\psi(\lambda) \quad ; \quad L^*\psi^*(\lambda) = \lambda\psi^*(\lambda) \;\; , \;\; \frac{\partial}{\partial t_l}\psi^*(\lambda) = - \left(L^* \right)^{\frac{l}{m}}_{(+)} \psi^*(\lambda) \tag{5}$$

$\psi^{(*)}(\lambda)$ is called (adjoint) Baker-Akhiezer (BA) function.

The BA function of the generic Lax operator L (1) is obtained from the BA function of the "free" Lax operator $L^{(0)} = D^m$:

$$\psi^{(0)}(\lambda) = \exp\Big\{ \sum_{l\geq 1} t_l \lambda^{\frac{l}{m}} \Big\} \equiv e^{\xi(\{t\},\lambda)} \tag{6}$$

[4] Here and below, the superscript "*" on operators indicates pseudo-differential operator conjugation.

by applying the dressing operator W (2) to (6) :

$$\psi(\lambda) = W e^{\xi(\{t\},\lambda)} = \frac{\tau(\{t_l - 1/l\lambda^{\frac{l}{m}}\})}{\tau(\{t_l\})} e^{\xi(\{t\},\lambda)} \tag{7}$$

The function $\tau(\{t_l\})$ of all evolution parameters is called <u>τ-function</u> of the (generalized) **KP** hierarchy and by itself constitutes an alternative natural way to describe the pertinent integrable system.

The **KP** system (1) is endowed with bi-Hamiltonian Poisson bracket structures (another expression of its integrability) which results from the two compatible Hamiltonian structures on the algebra of pseudo-differential operators $\Psi\mathcal{DO}$ [15]. The latter are given by:

$$\{\langle L|\,X\rangle\,,\,\langle L|\,Y\rangle\,\}_1 = -\langle L\;|\,[X,\,Y]\rangle \tag{8}$$

$$\begin{aligned}\{\langle L|\,X\rangle\,,\,\langle L|\,Y\rangle\,\}_2 &= \mathrm{Tr}_A\left((LX)_{(+)}\,LY - (XL)_{(+)}\,YL\right)\\ &+ \frac{1}{m}\int dx\,\mathrm{Res}\Big([\,L\,,\,X\,]\Big)\partial^{-1}\mathrm{Res}\Big([\,L\,,\,Y\,]\Big)\end{aligned} \tag{9}$$

where the following notations are used. $<\cdot|\cdot>$ denotes the standard bilinear pairing in $\Psi\mathcal{DO}$ via the Adler trace $\langle L|\,X\rangle = \mathrm{Tr}_A\,(LX)$ with $\mathrm{Tr}_A X = \int \mathrm{Res}X$. Here X, Y are arbitrary elements of the algebra of pseudo-differential operators of the form $X = \sum_{k\geq-\infty} D^k X_k$ and similarly for Y. The second term on the r.h.s. of (9) is a Dirac bracket term originating from the second-class Hamiltonian constraint $v_{m-1} = 0$ on L (1).

In terms of the Lax coefficient functions $v_{m-2}, \ldots, v_0, u_1, u_2, \ldots$, the first Poisson bracket structure (8) takes the form of an infinite-dimensional Lie algebra which is a direct sum of two subalgebras spanned by $\{v_j\}$ and $\{u_i\}$, respectively. The latter is called $\mathbf{W_{1+\infty}}$-algebra [5]. Its Cartan subalgebra contains the infinite set of (Poisson-)commuting **KP** integrals of motion $H_{l-1} = \frac{1}{l}\mathrm{Tr}_A L^{\frac{l}{m}}$ whose densities are expressed in terms of the τ-function (7) as:

$$\partial_x \frac{\partial}{\partial t_l} \ln\tau = \mathrm{Res}L^{\frac{l}{m}} \tag{10}$$

In turn, the second Poisson bracket structure (9) spans a nonlinear (quadratic) algebra called $\hat{\mathbf{W}}_\infty(m)$ [16], which is an infinite-dimensional generalization of Zamolodchikov's W_N conformal algebras [6].

3. Constrained KP Hierarchies. Free-Field Realizations

Let us now turn our attention to a specific class of Hamiltonian reductions of the full (generalized) **KP** system (1) (cKP hierarchies, for short), where the purely pseudo-differential part of the **KP** Lax operator is parametrized through a *finite* number of functions (fields). To this end let us recall the notion of (adjoint) eigenfunction of a Lax operator (4). As in [17] let us consider the flow of a vector field ∂_α given by:

$$\partial_\alpha L = \Big[\,L\,,\,\sum_{i=1}^{M}\Phi_i D^{-1}\Psi_i\,\Big] \tag{11}$$

where Φ_i, Ψ_i are a set of M independent (adjoint) eigenfunctions of L (1). Using the simple identity valid for any differential operator $B_{(+)}$:

$$\Big[\,B_{(+)}\,,\,\Phi_i D^{-1}\Psi_i\,\Big]_- = \big(B_{(+)}\Phi_i\big)\,D^{-1}\Psi_i - \Phi_i D^{-1}\left(B^*_{(+)}\Psi_i\right) \tag{12}$$

one can easily show that:

$$\left[\partial_\alpha, \frac{\partial}{\partial t_l}\right] L = 0 \qquad , \quad l = 1, 2, \ldots \tag{13}$$

Now, the constrained KP hierarchy (denoted as $\mathsf{cKP}_{m,M}$) is obtained by identifying the "ghost" flow ∂_α with the isospectral flow $\frac{\partial}{\partial t_m}$ which, upon comparison of (11) with (1), implies the following constrained form of L:

$$L \equiv L_{m,M} = L_{(+)} + \sum_{i=1}^{M}\Phi_i D^{-1}\Psi_i = D^m + \sum_{j=0}^{m-2} v_j D^j + \sum_{i=1}^{M}\Phi_i D^{-1}\Psi_i \tag{14}$$

subject to the same Lax evolution equations as in (1). Moreover, using again identity (12) one finds that the functions Φ_i, Ψ_i remain (adjoint) eigenfunctions of the constrained Lax operator $L_{m,M}$ (14).

As shown in ref.[18], the $\mathsf{cKP}_{m,M}$ hierarchies given by (14) are equivalent to the so called "multi-boson" cKP hierarchies [19]:

$$L_{m,M} = L_{(+)} + \sum_{i=1}^{M} A_i^{(M)} \left(D - B_i^{(M)}\right)^{-1} \left(D - B_{i+1}^{(M)}\right)^{-1} \cdots \left(D - B_M^{(M)}\right)^{-1} \tag{15}$$

$$A_k^{(M)} = (-1)^{M-k} \sum_{s=1}^{k} \Phi_s \frac{W\left[\Psi_M, \ldots, \Psi_{k+1}, \Psi_s\right]}{W\left[\Psi_M, \ldots, \Psi_{k+1}\right]} \tag{16}$$

$$B_k^{(M)} = -\partial_x \ln \frac{W\left[\Psi_M, \ldots, \Psi_{k+1}, \Psi_k\right]}{W\left[\Psi_M, \ldots, \Psi_{k+1}\right]} \tag{17}$$

where

$$W\left[f_1, \ldots, f_k\right] \equiv \det \left\| \partial_x^{i-1} f_j \right\| \tag{18}$$

denotes the standard Wronskian.

There is still another useful representation of the $\mathsf{cKP}_{m,M}$ Lax operator as a ratio of two purely differential Lax operators [20, 21, 22]:

$$L_{m,M} = L_{m+M} \left(L_M\right)^{-1} \quad ; \quad m, M \geq 1 \tag{19}$$

$$L_{m+M} \equiv \left(D - b_{m+M}\right)\left(D - b_{m+M-1}\right) \cdots \left(D - b_1\right) \quad , \quad L_M \equiv \left(D - \tilde{b}_M\right)\left(D - \tilde{b}_{M-1}\right) \cdots \left(D - \tilde{b}_1\right) \tag{20}$$

where the coefficients $b_j, \tilde{b}_j$ are subject to the constraint:

$$\sum_{j=1}^{m+M} b_j - \sum_{l=1}^{M} \tilde{b}_l = 0 \tag{21}$$

As already proved in detail in ref.[21], the $\mathsf{cKP}_{m,M}$ Lax operator $L = L_{m,M}$ obeys the same two compatible Poisson bracket structures (8) and (9), *i.e.*, $\mathsf{cKP}_{m,M}$ hierarchies are legitimate Hamiltonian reductions of the full (generalized) **KP** hierarchy. Moreover, the second Poisson bracket structure (9) in terms of the coefficients (20) takes the form of free-field Poisson bracket algebra:

$$\begin{aligned}
\left\{ b_i(x)\,,\, b_j(y) \right\} &= \left(\delta_{ij} - \frac{1}{m}\right) \delta'(x-y)\,, \qquad i,j = 1, \ldots, m+M \\
\left\{ \tilde{b}_k(x)\,,\, \tilde{b}_l(y) \right\} &= -\left(\delta_{kl} + \frac{1}{m}\right) \delta'(x-y)\,, \qquad k,l = 1, \ldots, M \\
\left\{ b_i(x)\,,\, \tilde{b}_l(y) \right\} &= \frac{1}{m} \delta'(x-y)
\end{aligned} \tag{22}$$

which, as demonstrated in refs.[23, 21], is precisely the Cartan subalgebra of the graded $SL(m+M, M)$ Kac-Moody algebra. This latter property justifies the alternative name of the constrained $\mathsf{cKP}_{m,M}$ hierarchies – $SL(m+M, M)$ KP-KdV hierarchies.

In other words, (19)–(22) provide via eq.(9) explicit free-field realizations of the nonlinear $\hat{\mathbf{W}}_\infty(m)$ algebra. Similar free-field realizations exist also for $\mathbf{W_{1+\infty}}$ – the first **KP** Poisson bracket structure (see refs.[24, 21]).

4. Additional Symmetries and Darboux-Bäcklund Transformations

Let L be again a pseudo-differential Lax operator of the full generalized KP hierarchy (1) (recall $x \equiv t_1$) and let M be a pseudo-differential operator "canonically conjugated" to L such that:

$$\left[L\,,\, M \right] = 1\!\!1 \quad , \quad \frac{\partial}{\partial t_l} M = \left[L_{(+)}^{\frac{l}{m}}\,,\, M \right] \tag{23}$$

Within the Sato-Wilson dressing operator formalism (2)–(3) the M-operator can be expressed in terms of dressing of the "bare" $M^{(0)}$ operator:

$$M^{(0)} = \sum_{l \geq 1} \frac{l}{m} t_l D^{l-m} = X_{(m)} + \sum_{l \geq 1} \frac{l+m}{m} t_{m+l} D^l \tag{24}$$

$$X_{(m)} \equiv \sum_{l=1}^{m} \frac{l}{m} t_l D^{l-m} \tag{25}$$

conjugated to the "bare" Lax operator $L^{(0)} = D^m$, *i.e.*:

$$M = WM^{(0)}W^{-1} = WX_{(m)}W^{-1} + \sum_{l\geq 1}\frac{l+m}{m}t_{m+l}L^{\frac{l}{m}} = \sum_{l\geq 0}\frac{l+m}{m}t_{m+l}L^{\frac{l}{m}}_{(+)} + M_- \tag{26}$$

$$M_- = WX_{(m)}W^{-1} - t_m - \sum_{l\geq 1}\frac{l+m}{m}t_{m+l}\frac{\partial}{\partial t_l}W\,.\,W^{-1} \tag{27}$$

where in (27) we used eqs.(3). Note that $X_{(m)}$ is a pseudo-differential operator satisfying $\Big[\,D^m\,,\,X_{(m)}\Big] = \mathbb{1}$.

On BA functions (5) the action of M is as follows:

$$M\psi(\lambda) = \left(\frac{\partial}{\partial\lambda} + \alpha_m(\lambda)\right)\psi(\lambda) \tag{28}$$

where $\alpha_m(\lambda)$ is a function of λ only[5].

Since any eigenfunction Φ (4) can be represented as a linear "superposition" of BA functions (5) :

$$\Phi(\{t\}) = \int_\Gamma d\lambda\,\phi(\lambda)\psi(\lambda,\{t\}) \tag{29}$$

(with an appropriate contour Γ in the complex λ-plane, such that the integral in (29) exists), eq.(28) implies that:

$$M\Phi(\{t\}) = \int_\Gamma d\lambda\left(-\frac{\partial}{\partial\lambda} + \alpha_m(\lambda)\right)\phi(\lambda)\psi(\lambda,\{t\}) \tag{30}$$

The so called *additional (non-isospectral) symmetries* [13, 11] are defined as vector fields on the space of **KP** Lax operators (1) or, alternatively, on the dressing operator (2), through their flows as follows:

$$\bar{\partial}_{k,n}L = -\Big[\left(L^kM^n\right)_-\,,\,L\Big] = \Big[\left(L^kM^n\right)_{(+)}\,,\,L\Big] + nL^kM^{n-1} \quad , \quad \bar{\partial}_{k,n}W = -\left(L^kM^n\right)_-W \tag{31}$$

which *commute* with the usual **KP** flows $\frac{\partial}{\partial t_l}$ (1).

Let us now turn our attention to the notion of Darboux-Bäcklund (DB) transformations of (generalized) **KP** hierarchy (1) and its reductions – $\mathsf{cKP}_{m,M}$ hierarchies (14), defined as follows [14, 18] :

$$\widetilde{L} = TLT^{-1} \equiv \widetilde{L}_{(+)} + \widetilde{L}_- \quad , \quad T \equiv \chi D\chi^{-1} \tag{32}$$

$$\widetilde{L}_{(+)} = L_{(+)} + \chi\left(\partial_x\left(\chi^{-1}L_{(+)}\chi\right)_{\geq 1}D^{-1}\right)\chi^{-1} \tag{33}$$

$$\widetilde{L}_- = \widetilde{\Phi}_0D^{-1}\widetilde{\Psi}_0 + \chi D\chi^{-1}L_-\chi D^{-1}\chi^{-1} \quad \left(= \widetilde{\Phi}_0D^{-1}\widetilde{\Psi}_0 + \sum_{i=1}^M\widetilde{\Phi}_iD^{-1}\widetilde{\Psi}_i \quad \text{for} \;\; L = L_{m,M}\right) \tag{34}$$

$$\widetilde{\Phi}_0 = \left(\chi D\chi^{-1}L\right)\chi \equiv TL\chi \quad , \quad \widetilde{\Psi}_0 = \chi^{-1} \tag{35}$$

$$\widetilde{\Phi}_i = \chi\partial_x\left(\chi^{-1}\Phi_i\right) \quad , \quad \widetilde{\Psi}_i = -\chi^{-1}\partial_x^{-1}\left(\Psi_i\chi\right) \tag{36}$$

where χ is an (non-BA) eigenfunction of L. The DB-transformed Lax operator (32) satisfies the same flow equations w.r.t. t_l as in (1): $\frac{\partial}{\partial t_l}\widetilde{L} = \Big[\,\widetilde{L}^{\frac{l}{m}}_{(+)}\,,\,\widetilde{L}\,\Big]$ due to the simple identity valid for any pseudo-differential operator B

$$\left(\chi D\chi^{-1}B\chi D^{-1}\chi^{-1}\right)_{(+)} = \chi D\chi^{-1}B_{(+)}\chi D^{-1}\chi^{-1} - \chi\partial_x\left(\chi^{-1}(B_{(+)}\chi)\right)D^{-1}\chi^{-1} \tag{37}$$

(and using the fact that χ is an eigenfunction of L). Moreover, eq.(34) shows that, in order to preserve the $\mathsf{cKP}_{m,M}$ form of the DB-transformed Lax operator $\widetilde{L} = \widetilde{L}_{m,M}$, we have to choose $\chi = \Phi_{i_0}$ where Φ_{i_0} is any one of the eigenfunctions of the initial $L = L_{m,M}$ (14).

One can generalize (32)–(36) for successive Darboux-Bäcklund transformations on the initial $L = L_{m,M} \equiv L^{(0)}$ as follows. Within each subset of M successive steps we can perform the DB transformations w.r.t. the M

[5]The appearance of $\alpha_m(\lambda)$ can be traced back to the ambiguity in the definition of the dressing operator (2): $W \longrightarrow WW_0$ where $W_0 = 1 + \sum_{i\geq 1}c_iD^{-i}$ with *constant* coefficients c_i.

different eigenfunctions of (14). Repeated use of the following important composition formula for Wronskians [25] :

$$T_k\, T_{k-1} \cdots T_1(f) \;=\; \frac{W_k(f)}{W_k} \tag{38}$$

where

$$T_j = \frac{W_j}{W_{j-1}} D \frac{W_{j-1}}{W_j} = \left(D + \left(\ln \frac{W_{j-1}}{W_j} \right)' \right) \quad ; \quad W_0 = 1 \tag{39}$$

$$W_k \equiv W_k[\psi_1, \ldots, \psi_k] \quad , \quad W_{k-1}\,(f) \equiv W_k[\psi_1, \ldots, \psi_{k-1}, f] \tag{40}$$

and employing short-hand notations:

$$T_i^{(k)} \equiv \Phi_i^{(k)} D \left(\Phi_i^{(k)} \right)^{-1} \qquad ; \qquad \chi_i^{(s)} \equiv \left(L^{(0)} \right)^s \Phi_i^{(0)} \quad , \; i = 1, \ldots, M \tag{41}$$

where the upper indices in brackets indicate the order of the corresponding DB step, yields the following generalization of (35)–(36) (below $1 \leq l \leq M$) :

$$\begin{aligned} &\Phi_i^{(kM+l)} = T_l^{(kM+l-1)} \ldots T_1^{(kM)} T_M^{(kM-1)} \ldots T_1^{((k-1)M)} \ldots T_M^{(M-1)} \ldots T_1^{(0)} \chi_i^{(k\pm)} \\ &= \frac{W\left[\Phi_1^{(0)}, \ldots, \Phi_M^{(0)}, \chi_1^{(1)}, \ldots, \chi_M^{(1)}, \ldots, \chi_1^{(k-1)}, \ldots, \chi_M^{(k-1)}, \chi_1^{(k)}, \ldots, \chi_l^{(k)}, \chi_i^{(k\pm)} \right]}{W\left[\Phi_1^{(0)}, \ldots, \Phi_M^{(0)}, \chi_1^{(1)}, \ldots, \chi_M^{(1)}, \ldots, \chi_1^{(k-1)}, \ldots, \chi_M^{(k-1)}, \chi_1^{(k)}, \ldots, \chi_l^{(k)} \right]} \\ &\chi_i^{(k_+)} \equiv \chi_i^{(k+1)} \quad \text{for } \; 1 \leq i \leq l \qquad ; \qquad \chi_i^{(k_-)} \equiv \chi_i^{(k)} \quad \text{for } \; l+1 \leq i \leq M \end{aligned} \tag{42}$$

Correspondingly, for the τ function (7) after $km + l$ steps of successive DB transformations we get:

$$\begin{aligned} \frac{\tau^{(kM+l)}}{\tau^{(0)}} &= \Phi_l^{(kM+l-1)} \ldots \Phi_1^{(kM)} \Phi_M^{(kM-1)} \ldots \Phi_1^{((k-1)M)} \ldots \Phi_M^{(M-1)} \ldots \Phi_1^{(0)} \\ &= W\left[\Phi_1^{(0)}, \ldots, \Phi_M^{(0)}, \chi_1^{(1)}, \ldots, \chi_M^{(1)}, \ldots, \chi_1^{(k-1)}, \ldots, \chi_M^{(k-1)}, \chi_1^{(k)}, \ldots, \chi_l^{(k)} \right] \end{aligned} \tag{43}$$

We now formulate the main result of this section – the condition for *compatibility* between additional-symmetry flows (31) and Darboux-Bäcklund transformations (32).

Let Φ be an eigenfunction of L defining a Darboux-Bäcklund transformation, *i.e.* :

$$\frac{\partial}{\partial t_l} \Phi = L_{(+)}^{\frac{l}{m}} \Phi \quad , \quad \widetilde{L} = \left(\Phi D \Phi^{-1} \right) \, L \, \left(\Phi D^{-1} \Phi^{-1} \right) \tag{44}$$

or, in terms of dressing operator:

$$\widetilde{W} = \left(\Phi D \Phi^{-1} \right) \, W \, D^{-1} \tag{45}$$

Then the DB-transformed M operator (cf. (26)) acquires the form:

$$\widetilde{M} = \left(\Phi D \Phi^{-1} \right) \, M \, \left(\Phi D^{-1} \Phi^{-1} \right) = \sum_{l \geq 0} \frac{l+m}{m} t_{m+l} \widetilde{L}_{(+)}^{\frac{l}{m}} + \widetilde{M}_- \tag{46}$$

$$\widetilde{M}_- = \widetilde{W} \widetilde{X}_{(m)} \widetilde{W}^{-1} - t_m - \sum_{l \geq 1} \frac{l+m}{m} t_{m+l} \frac{\partial}{\partial t_l} \widetilde{W} \, . \, \widetilde{W}^{-1} \tag{47}$$

where $\widetilde{X}_{(m)} = D X_{(m)} D^{-1}$ with $X_{(m)}$ as in (25). Clearly $\widetilde{X}_{(m)}$, like $X_{(m)}$, is also admissible as canonically conjugated to D^m.

The DB-transformed BA function reads in accordance with (45):

$$\widetilde{\psi}(\lambda) = \lambda^{-\frac{1}{m}} \Phi \partial_x \left(\Phi^{-1} \psi(\lambda) \right) \tag{48}$$

and the DB-transformed M-operator acts on it as:

$$\widetilde{M} \widetilde{\psi}(\lambda) = \left(\frac{\partial}{\partial \lambda} + \widetilde{\alpha}_m(\lambda) \right) \widetilde{\psi}(\lambda) \quad , \quad \widetilde{\alpha}_m(\lambda) = \alpha_m(\lambda) + \frac{1}{m} \lambda^{-1} \tag{49}$$

Taking into account (44), we arrive at the following important

Proposition. *Additional symmetry flows (31) commute with Darboux-Bäcklund transformations (44)–(46), i.e.*

$$\bar{\partial}_{k,n}\widetilde{L} = -\Big[\left(\widetilde{L}^k\widetilde{M}^n\right)_-\,,\,\widetilde{L}\Big] \qquad , \qquad \bar{\partial}_{k,n}\widetilde{W} = -\left(\widetilde{L}^k\widetilde{M}^n\right)_-\widetilde{W} \tag{50}$$

if and only if the DB-generating *eigenfunction* Φ *transforms under the additional symmetries as:*

$$\bar{\partial}_{k,n}\Phi = \left(L^k M^n\right)_{(+)}\Phi \tag{51}$$

Motivated by applications to (multi-)matrix models (see next sections and ref.[30]), one can require invariance under some of the additional-symmetry flows, *e.g.*, under the lowest one $\bar{\partial}_{0,1}$ known as "string-equation" constraint in the context of (multi-)matrix models:

$$\bar{\partial}_{0,1}L = 0 \qquad \to \qquad \Big[M_{(+)}\,,\,L\Big] = -1\!\!1 \qquad ; \qquad \bar{\partial}_{0,1}\Phi = 0 \qquad \to \qquad M_{(+)}\Phi = 0 \tag{52}$$

Eqs.(52), using second eq.(23),(26)–(29) and (51), lead to the following constraints for L, the BA function $\psi(\lambda)$ and the DB-generating eigenfunction Φ of L, respectively:

$$\sum_{l\geq 1}\frac{l+m}{m}t_{m+l}\frac{\partial}{\partial t_l}L + \Big[t_1\,,\,L\Big]\,\delta_{m,1} = -1\!\!1 \tag{53}$$

$$\left(\sum_{l\geq 1}\frac{l+m}{m}t_{m+l}\frac{\partial}{\partial t_l} + t_m - \alpha_m(\lambda)\right)\psi(\lambda) = \frac{\partial}{\partial\lambda}\psi(\lambda) \tag{54}$$

$$\left(\sum_{l\geq 1}\frac{l+m}{m}t_{m+l}\frac{\partial}{\partial t_l} + t_m\right)\Phi = 0 \qquad \to \qquad \Phi(\{t\}) = \int_\Gamma d\lambda\, e^{\int\alpha(\lambda)}\psi(\lambda;\{t\}) \tag{55}$$

Now let us recall the formula (43) for the τ-function ratio for $\mathsf{cKP}_{m,M}$ hierarchies subject to successive DB transformations. Noticing that the eigenfunctions $\Phi^{(k)}$ of the DB-transformed Lax operators $L^{(k)}$ satisfy the *same* constraint eq.(55) irrespective of the DB-step k, we arrive at the following result ("string-equation" constraint on the τ-functions) :

Proposition. *The Wronskian τ-functions (43) of the* $\mathsf{cKP}_{m,M}$ *hierarchies satisfy the constraint equation:*

$$\left(\sum_{l\geq 1}\frac{l+m}{m}t_{m+l}\frac{\partial}{\partial t_l} + nt_m\right)\frac{\tau^{(n)}}{\tau^{(0)}} = 0 \tag{56}$$

Example: It is well-known that the discrete one-matrix model can be associated to the following chain of the Lax operators connected via DB transformations:

$$L^{(k+1)} = \left(\Phi^{(k)}D\Phi^{(k)^{-1}}\right)\,L^{(k)}\,\left(\Phi^{(k)}D^{-1}\Phi^{(k)^{-1}}\right) = D + \Phi^{(k+1)}D^{-1}\Psi^{(k+1)} \tag{57}$$

$$\Phi^{(k+1)} = \Phi^{(k)}\left(\ln\Phi^{(k)}\right)'' + \left(\Phi^{(k)}\right)^2\Psi^{(k)} \quad , \quad \Psi^{(k+1)} = \left(\Phi^{(k)}\right)^{-1} \tag{58}$$

where

$$\Phi^{(n)} = \frac{W_{n+1}[\phi,\partial\phi,\ldots,\partial^n\phi]}{W_n[\phi,\partial\phi,\ldots,\partial^{n-1}\phi]} \qquad , \qquad \frac{\tau^{(n)}}{\tau^{(0)}} = W_n[\phi,\partial\phi,\ldots,\partial^{n-1}\phi] \tag{59}$$

with $\phi = \int d\lambda\exp\left(\sum_{k=1}^\infty t_k\lambda^k\right)$. The above proposition (with $m=1$) coincides perfectly with the "string-equation" $\mathcal{L}_{-1}^{(N)}W_N[\phi,\partial\phi,\ldots,\partial^{N-1}\phi] = 0$, with $\mathcal{L}_{-1}^{(N)} = \sum_{k=2}^\infty kt_k\frac{\partial}{\partial t_{k-1}} + Nt_1$.

5. Multi-Matrix Models as $\mathbf{cKP_{m,1}}$ Hierarchies

The partition function of the multi-matrix (q-matrix) string model reads:

$$Z_N[\{t^{(1)}\},\ldots,\{t^{(q)}\},\{g\}] = \int dM_1\ldots dM_q\exp-\left\{\sum_{\alpha=1}^q\sum_{r_\alpha=1}^{p_\alpha}t_{r_\alpha}^{(\alpha)}\,\mathrm{Tr}\,M_\alpha^{r_\alpha} + \sum_{\alpha=1}^{q-1}g_{\alpha,\alpha+1}\,\mathrm{Tr}\,M_\alpha M_{\alpha+1}\right\} \tag{60}$$

where M_α are Hermitian $N \times N$ matrices, and the orders of the matrix "potentials" p_α may be finite or infinite. In refs.[26] it was shown[6] that, via the method of generalized orthogonal polynomials [28], one associates to (60) generalized Toda-like lattice systems subject to specific constraints, so that Z_N and its derivatives w.r.t. the coupling parameters can be expressed in terms of solutions of the underlying Toda-like discrete integrable hierarchy where $\{t^{(1)}\}, \ldots, \{t^{(q)}\}$ play the role of "evolution" parameters. This Toda-like discrete integrable hierarchy differs from the full generalized Toda lattice hierarchy [29] in that the associated Toda matrices in the first hierarchy are semiinfinite and contain in general *finite* number of non-zero diagonals.

It turns out that, in order to identify the continuum cKP integrable hierarchy which provides the exact solution for (60), we need the following subset of the associated linear system and the corresponding Lax ("zero-curvature") representation from the Toda-like lattice system [26] :

$$Q(1)_{nm}\psi_m = \lambda\psi_n \quad , \quad \frac{\partial}{\partial t_r^{(1)}}\psi_n = -\left(Q(1)_-^r\right)_{nm}\psi_m \quad , \quad \frac{\partial}{\partial t_s^{(q)}}\psi_n = -\left(Q(q)_-^s\right)_{nm}\psi_m \tag{61}$$

$$\frac{\partial}{\partial t_r^{(1)}} Q(1) = \left[Q(1)_{(+)}^r, Q(1)\right] \quad , \quad \frac{\partial}{\partial t_s^{(q)}} Q(1) = \left[Q(1), Q(q)_-^s\right] \tag{62}$$

$$\frac{\partial}{\partial t_r^{(1)}} Q(q) = \left[Q_{(+)}^r, Q(q)\right] \quad , \quad \frac{\partial}{\partial t_s^{(q)}} Q(q) = \left[Q(q), Q(q)_-^s\right] \tag{63}$$

In what follows it is convenient to introduce the short-hand notations:

$$t_r \equiv t_r^{(1)} \quad , \quad r = 1, \ldots, p_1 \quad ; \quad \tilde{t}_s \equiv t_s^{(q)} \quad , \quad s = 1, \ldots, p_q \quad ; \quad Q \equiv Q(1) \quad , \quad \widetilde{Q} \equiv Q(q) \tag{64}$$

Further, there is a series of additional constraints ("coupling conditions") relating $Q \equiv Q(1)$ and $\widetilde{Q} \equiv Q(q)$. In the two-matrix model case ($q = 2$) their explicit form is:

$$-g\left[Q, \widetilde{Q}\right] = 1\!\!1 \tag{65}$$

$$Q_{(-)} = -\sum_{s=1}^{p_2-1} \frac{(s+1)}{g}\tilde{t}_{s+1}\widetilde{Q}_{(-)}^s - \frac{1}{g}\tilde{t}_1 1\!\!1 \tag{66}$$

$$\widetilde{Q}_{(+)} = -\sum_{r=1}^{p_1-1} \frac{(r+1)}{g} t_{r+1} Q_{(+)}^r - \frac{1}{g} t_1 1\!\!1 \tag{67}$$

Here the subscripts $-/+$ denote lower/upper triangular parts, whereas $(+)/(-)$ denote upper/lower triangular plus diagonal parts. In the higher ($q \geq 3$) multi-matrix case the "coupling conditions" have much more intricate form (involving also the "intermediate" $Q(2), \ldots, Q(q-1)$ matrices). However, their explicit form will not be needed to find the solution for Z_N (60) since we will be able to extract the relevant information solely from the discrete Lax system (62)–(63) and the relations expressing $Q \equiv Q(1)$, $\widetilde{Q} \equiv Q(q)$ in terms of orthogonal polynomial factors (see eqs.(71) below).

The parametrization for the matrix elements of the Jacobi matrices $Q \equiv Q(1)$ and $\widetilde{Q} \equiv Q(q)$ is as follows:

$$\begin{gathered} Q_{nn} \equiv Q(1)_{nn} = a_0(n) \quad , \quad Q(1)_{n,n+1} \equiv Q_{n,n+1} = 1 \\ Q(1)_{n,n-k} \equiv Q_{n,n-k} = a_k(n) \quad k = 1, \ldots, m(1) \quad , \quad m(1) = (p_q - 1)\ldots(p_2 - 1) \\ Q(1)_{nm} \equiv Q_{nm} = 0 \quad \text{for} \quad m - n \geq 2 \ , \ n - m \geq m(1) + 1 \end{gathered} \tag{68}$$

$$\begin{gathered} Q(q)_{nn} \equiv \widetilde{Q}_{nn} = b_0(n) \quad , \quad Q(q)_{n,n-1} \equiv \widetilde{Q}_{n,n-1} = R_n \\ Q(q)_{n,n+k} \equiv \widetilde{Q}_{n,n+k} = b_k(n) R_{n+1}^{-1} \ldots R_{n+k}^{-1} \quad k = 1, \ldots, m(q) \quad , \quad m(q) = (p_{q-1} - 1)\ldots(p_1 - 1) \\ Q(q)_{nm} \equiv \widetilde{Q}_{nm} = 0 \quad \text{for} \quad n - m \geq 2 \ , \ m - n \geq m(q) + 1 \end{gathered} \tag{69}$$

In terms of the $Q \equiv Q(1)$, $\widetilde{Q} \equiv Q(q)$ matrix elements the partition function (60) is expressed in the following way [26]:

$$Z_N = const \prod_{n=0}^{N-1} h_n \tag{70}$$

[6] See also refs.[27] and the lecture of Prof. L.Bonora in the present volume.

$$a_0(n)=\frac{\partial}{\partial t_1}\ln h_n \quad , \quad b_0(n)=\frac{\partial}{\partial \tilde{t}_1}\ln h_n \quad , \quad R_n=\frac{h_n}{h_{n-1}}$$

$$\frac{\partial}{\partial t_r}\ln h_n = Q^r_{nn} \quad , \quad \frac{\partial}{\partial \tilde{t}_s}\ln h_n = \widetilde{Q}^s_{nn} \tag{71}$$

$$\frac{\partial}{\partial t_1}\ln Z_N=\sum_{n=0}^{N-1} a_0(n) \quad , \quad \frac{\partial}{\partial \tilde{t}_1}\ln Z_N=\sum_{n=0}^{N-1} b_0(n)$$

$$\frac{\partial}{\partial t_r}\ln Z_N=\sum_{n=0}^{N-1} Q^r_{nn} \quad , \quad \frac{\partial}{\partial \tilde{t}_s}\ln Z_N=\sum_{n=0}^{N-1} \widetilde{Q}^s_{nn} \tag{72}$$

where h_n are the normalization factors in the nonlocally generalized orthogonal polynomial formalism [28] (using notations (64)) :

$$h_n\delta_{nm}=\int_\Gamma\int_\Gamma d\lambda d\mu\, P_n(\lambda)\exp\Bigl\{\sum_{r=1}^{p_1}\lambda^r t_r\Bigr\}\rho(\lambda,\mu;\{t''\},\{g\})\Bigl\{\sum_{s=1}^{p_q}\mu^s\tilde{t}_s\Bigr\}\widetilde{P}_m(\mu) \tag{73}$$

$$\rho(\lambda,\mu;\{t''\},\{g\})=\int_\Gamma\prod_{\alpha=2}^{q-1}d\nu_\alpha\ \exp\Bigl\{\sum_{\alpha=2}^{q-1}\sum_{r_\alpha=1}^{p_\alpha}t^{(r_\alpha)}_{r_\alpha}\nu_\alpha^{r_\alpha}+\sum_{\alpha=2}^{q-2}g_{\alpha,\alpha+1}\nu_\alpha\nu_{\alpha+1}+g_{12}\lambda\nu_2+g_{q-1,q}\nu_{q-1}\mu\Bigr\} \tag{74}$$

$$\{t''\}\equiv\Bigl(t^{(2)},\ldots,t^{(q-1)}\Bigr)$$

As in the case of two-matrix model [21, 18], using the lattice equations of motion (eqs.(62)–(63) for $r=1, s=1$) we obtain the following important:

Proposition. *The matrix elements of $Q\equiv Q(1)$ are completely expressed in terms of the matrix elements of $\widetilde{Q}\equiv Q(q)$ through the relations:*

$$Q_{(-)}=\sum_{s=0}^{m(1)}\alpha_s\widetilde{Q}^s_{(-)} \qquad , \qquad Q^{\frac{s}{m(1)}}_{(-)}=\sum_{\sigma=0}^{s}\gamma_{s\sigma}\widetilde{Q}^\sigma_{(-)} \quad , \quad s=0,1,\ldots,m(1) \tag{75}$$

where the coefficients $\alpha_0,\gamma_{s,0}$ are t_1-independent, whereas the coefficients $\alpha_s,\gamma_{s\sigma}$ with $\sigma\geq 1$ are independent of $t_1,\tilde{t}_1$. All $\gamma_{s\sigma}$ are expressed through $\alpha_s\equiv\gamma_{m(1),s}$ solely:

$$\gamma_{ss}=(\gamma_{11})^s \ , \ \gamma_{s,s-1}=s\,(\gamma_{11})^{s-1}\gamma_{10} \ , \ \gamma_{s,s-2}=(\gamma_{11})^{s-2}\left[\frac{s(s-1)}{2}(\gamma_{10})^2+s\left(\frac{\gamma_{31}}{3\gamma_{11}}-\gamma_{10}^2\right)\right] \tag{76}$$

$$\gamma_{11}=\bigl(\alpha_{m(1)}\bigr)^{\frac{1}{m(1)}} \quad , \quad \gamma_{10}=\frac{\alpha_{m(1)-1}}{m(1)\,\bigl(\alpha_{m(1)}\bigr)^{\frac{m(1)-1}{m(1)}}}$$

$$\frac{\gamma_{31}}{3\gamma_{11}}-\gamma_{10}^2=\frac{1}{m(1)}\left[\frac{\alpha_{m(1)-2}}{\bigl(\alpha_{m(1)}\bigr)^{\frac{m(1)-2}{m(1)}}}-\frac{m(1)-1}{2m(1)}\frac{\alpha^2_{m(1)-1}}{\bigl(\alpha_{m(1)}\bigr)^{\frac{2(m(1)-1)}{m(1)}}}\right] \tag{77}$$

In the two-matrix model the explicit form of the coefficients α_s reads: $\alpha_s=-\frac{(s+1)}{g}\tilde{t}_{s+1}$.

Similarly, we have the dual statement with the rôles of $Q\equiv Q(1)$ and $\widetilde{Q}\equiv Q(q)$ interchanged.

As an important consequence of (75), let us take its diagonal 00-part and use the last eq.(71) which yields:

$$\frac{\partial}{\partial t_1}h_0=\left(\sum_{s=1}^{m(1)}\alpha_s\frac{\partial^s}{\partial\tilde{t}_1^{\,s}}+\alpha_0\right)h_0 \tag{78}$$

This equation is the only remnant of the constraints ("coupling conditions") on the multi-matrix model Q-matrices which will be used in the sequel.

Based on our experience with the two-matrix model [21, 18], it turns out natural to introduce the fractional power of $Q\equiv Q(1)$:

$$\widehat{Q}=Q^{\frac{1}{m(1)}}\equiv Q(1)^{\frac{1}{m(1)}} \tag{79}$$

whose parametrization closely resembles that of $\widetilde{Q} \equiv Q(q)$ (69) :

$$\widehat{Q}_{nn} = \widehat{b}_0(n) \quad , \quad \widehat{Q}_{n,n-1} = \widehat{R}_n \quad , \quad \widehat{Q}_{n,n+k} = \widehat{b}_k(n)\widehat{R}^{-1}_{n+1} \ldots \widehat{R}^{-1}_{n+k} \quad k \geq 1$$
$$\widehat{Q}_{nm} = 0 \quad \text{for} \quad n - m \geq 2 \tag{80}$$

From eqs.(75) we find the following relation between the matrix elements of $\widehat{Q} \equiv Q(1)^{\frac{1}{m(1)}}$ and $\widetilde{Q}$:

$$\hat{R}_n = \gamma_{11} R_n \quad , \quad \hat{b}_0(n) = \gamma_{11} b_0(n) + \gamma_{10} \quad , \quad \hat{b}_1(n) = \gamma_{11}^2 b_1(n) + \frac{\gamma_{31}}{3\gamma_{11}} - \gamma_{10}^2 \tag{81}$$

etc., with γ-coefficients as in (76)–(77).

In order to identify the continuum cKP hierarchy associated with the general q-matrix model, as a first step we reexpress, using (75), the Toda-like lattice hierarchy (61)–(63) as a single set of flow equations for $\widehat{Q} \equiv Q(1)^{\frac{1}{m(1)}}$:

$$\widehat{Q}^{m(1)}_{nm} \psi_m = \lambda \psi_n \quad , \quad \frac{\partial}{\partial \hat{t}_s} \psi_n = -\left(\widehat{Q}^s_{(-)}\right)_{nm} \psi_m \tag{82}$$

$$\frac{\partial}{\partial \hat{t}_s} \widehat{Q} = \left[\widehat{Q}, \widehat{Q}^s_-\right] \quad , \quad s = 1, \ldots, p_q, 2m(1), 3m(1), \ldots, p_1 m(1) \tag{83}$$

$$t_r \equiv \hat{t}_{rm(1)} \quad \text{for} \quad r = 1, \ldots, p_1 \tag{84}$$

Here, as in the two-matrix case [21, 18], we have introduced a new subset of evolution parameters $\left\{\hat{t}_s\right\}$ instead of $\left\{\tilde{t}_s \equiv t^{(q)}_s\right\}$ defined as:

$$\frac{\partial}{\partial \hat{t}_s} = \sum_{\sigma=1}^{s} \gamma_{s\sigma} \frac{\partial}{\partial \tilde{t}_\sigma} \quad , \quad s = 1, \ldots, m(q) \tag{85}$$

with the same $\gamma_{s\sigma}$ as in (75). As a second step, one employs the Bonora-Xiong procedure [26] to get from the discrete Lax system (82)–(83) an equivalent continuum Lax system associated with a fixed lattice site n, where the continuum space coordinate is $x \equiv \hat{t}_1$. Namely, the latter continuum integrable system is obtained by writing eqs.(82) in more detail using the parametrization (68),(80) :

$$\lambda \psi_n = \psi_{n+1} + a_0(n)\psi_n + \sum_{k=1}^{p_2-1} a_k(n)\psi_{n-k} \tag{86}$$

$$\frac{\partial}{\partial \hat{t}_1} \psi_n = -\widehat{R}_n \psi_{n-1} \tag{87}$$

and further using (87) to express $\psi_{n\pm\ell}$ in terms of ψ_n at a fixed lattice site n in eq.(86) and the higher evolution eqs.(82) (for $s \geq 2$). Upon operator conjugation and an appropriate similarity transformation, it acquires the form (as before $x \equiv \hat{t}_1$) :

$$\frac{\partial}{\partial \hat{t}_s} L(n) = \left[\left(L^{\frac{s}{m(1)}}(n)\right)_{(+)}, L(n) \right] \quad , \quad s = 1, \ldots, p_q, 2m(1), 3m(1), \ldots, p_1 m(1) \tag{88}$$

$$L(n) = D_x^{m(1)} + m(1)\widehat{b}_1(n) D_x^{m(1)-2} + \cdots + \widehat{R}_{n+1} \left(D_x - \widehat{b}_0(n)\right)^{-1} \tag{89}$$

where $\widehat{b}_{0,1}(n), \widehat{R}_{n+1}$ are the matrix elements of $\widehat{Q}$ (80),(81). Rewriting (89) in the equivalent "eigenfunction" form:

$$L(n) = D_x^{m(1)} + m(1)\widehat{b}_1(n) D_x^{m(1)-2} + \cdots + \Phi(n+1) D_x^{-1} \Psi(n+1) \tag{90}$$

$$\Phi(n+1) \equiv \widehat{R}_{n+1} \exp\left\{\int \widehat{b}_0(n)\right\} \quad , \quad \Psi(n+1) \equiv \exp\left\{-\int \widehat{b}_0(n)\right\} \tag{91}$$

and comparing with (14), we identify the continuum integrable hierarchy (88), describing equivalently the discrete multi-matrix model, as a constrained cKP $_{m(1),1}$ hierarchy.

6. Partition Functions of Multi-Matrix Models: Darboux-Bäcklund Solutions [7]

[7] A more detailed presentation of the material in this section will appear in ref.[30].

Exactly as in the two-matrix case [21, 18], lattice shifts $n \to n+1$ in the underlying discrete Toda lattice system, described by (61)–(91), generate Darboux-Bäcklund transformations in the continuum $\mathsf{cKP}_{\,m(1),1}$ hierarchy (88)–(89). This is due to the fact that the latter continuum hierarchy preserves its form for any value of the discrete label n. The solutions for the eigenfunctions and τ-functions at each successive step of Darboux-Bäcklund transformation is given explicitly, as particular cases of eqs.(42)–(43), by:

$$\Phi(n) = \frac{W_{n+1}\left[\Phi(0), L(-1)\Phi(0), \ldots, L(-1)^n\Phi(0)\right]}{W_n\left[\Phi(0), L(-1)\Phi(0), \ldots, L(-1)^{n-1}\Phi(0)\right]} \tag{92}$$

$$\frac{\tau(n)}{\tau(-1)} = \prod_{j=0}^{n} \Phi(j) = W_{n+1}\left[\Phi(0), L(-1)\Phi(0), \ldots, L(-1)^n\Phi(0)\right] \tag{93}$$

$$\frac{\partial}{\partial \hat{t}_s}\Phi(0) = \left(L(-1)\right)_{(+)}^{\frac{s}{p_2-1}}\Phi(0) \quad , \quad s = 1, \ldots, p_2, 2(p_2-1), 3(p_2-1), \ldots, p_1(p_2-1) \tag{94}$$

where everything is expressed in terms of the eigenfunction $\Phi(0)$ of the "initial" Lax operator $L(-1)$. The difference with the two-matrix case is only the explicit form of the latter (recall $x \equiv \hat{t}_1$) :

$$L(-1) = e^{\gamma_{10}\hat{t}_1}\left(\sum_{s=0}^{m(1)} \alpha_s \gamma_{11}^{-s} D^s\right) e^{-\gamma_{10}\hat{t}_1} \tag{95}$$

where the coefficients $\alpha_s, \gamma_{10}, \gamma_{11}$ have more complicated dependence on $\{\hat{t}_s\}$ than in the two-matrix case.

Exactly as in the two-matrix case, we obtain the relation between the n-th step DB eigenfunction $\Phi(n)$ and the orthogonal polynomial normalization factor h_n (73) which generalizes (96) :

$$\Phi(n) \equiv e^{\int \widehat{b}_0(n)} = h_n\,\gamma_{11}^n\, \exp\left\{\hat{t}_1\gamma_{10} + \varepsilon(\hat{t}')\right\} \qquad , \quad \hat{t}' \equiv \left(\hat{t}_2, \ldots \hat{t}_{m(q)}\right) \tag{96}$$

Substituting (96) into (70) and using the Wronskian formula (93) we get:

$$Z_N = \prod_{n=0}^{N-1} h_n = \det\left\|\frac{\partial^{i-1}}{\partial \hat{t}_1^{\,i-1}}\left(L(-1)\right)^{j-1}\Phi(0)\right\| e^{-N\left(\hat{t}_1\gamma_{10}+\varepsilon(\hat{t}')\right)}\gamma_{11}^{-\frac{N(N-1)}{2}} \tag{97}$$

$$= \det\left\|\frac{\partial^{i-1}}{\partial \tilde{t}_1^{\,i-1}}\left(e^{-\hat{t}_1\gamma_{10}}L(-1)e^{\hat{t}_1\gamma_{10}}\right)^{j-1} h_0\right\| \tag{98}$$

where we absorbed the γ_{11}-factors via changing $\frac{\partial}{\partial \hat{t}_1} \to \frac{\partial}{\partial \tilde{t}_1}$ by the definition (85), *i.e.*, $\gamma_{11}^{-1}\frac{\partial}{\partial \hat{t}_1} = \frac{\partial}{\partial \tilde{t}_1}$. Now, we find using (95) and (78) :

$$\left(e^{-\hat{t}_1\gamma_{10}}L(-1)e^{\hat{t}_1\gamma_{10}}\right)^{j_1} h_0 = \left(\sum_{s=0}^{m(1)} \alpha_s \frac{\partial^s}{\partial \tilde{t}_1^{\,s}}\right)^{j-1} h_0 = \frac{\partial^{j-1}}{\partial t_1^{\,j-1}} h_0 \tag{99}$$

Substituting (99) into (98) yields the final result for the multi-matrix model partition function:

$$Z_N = \det\left\|\frac{\partial^{i+j-2} h_0}{\partial \tilde{t}_1^{\,i-1}\partial t_1^{\,j-1}}\right\| \tag{100}$$

which is functionally the same as for the two-matrix model, however, with a more complicated expression for h_0 (73) :

$$h_0 = \int_\Gamma\int_\Gamma d\lambda d\mu\, \exp\Bigl\{\sum_{r=1}^{p_1}\lambda^r t_r\Bigr\}\rho(\lambda,\mu;\{t''\},\{g\})\Bigl\{\sum_{s=1}^{p_q}\mu^s\tilde{t}_s\Bigr\} \tag{101}$$

Eq.(100) was previously obtained (see refs.[31]) from a different approach.

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